

Another Look at AR(1)

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ABSTRACT. Given a stationary first-order autoregressive process X_t (with lag-one correlation ρ satisfying $|\rho| < 1$), we examine the Central Limit Theorem for $\frac{1}{n} \ln |X_1 \cdots X_n|$ and compute variances to high precision. Given a nonstationary process X_t (with $|\rho| > 1$), we examine instead $\frac{1}{n} \ln |X_n|$ and study the distribution of $\ln |X_n| - n \ln |\rho|$.

This research began as an effort to better understand Viswanath's random integer recurrence [1]:

$$X_t = X_{t-1} \pm X_{t-2}, \quad X_0 = 1, \quad X_1 = 1,$$

$$\frac{1}{n} \ln |X_n| \rightarrow \ln(1.13198824\dots) \quad \text{almost surely as } n \rightarrow \infty$$

and one of Wright & Trefethen's real recurrences [2]:

$$X_t = X_{t-1} + \varepsilon_t X_{t-2}, \quad X_0 = 1, \quad X_1 = 1,$$

$$\frac{1}{n} \ln |X_n| \rightarrow \ln(1.057473553704\dots) \quad \text{almost surely as } n \rightarrow \infty$$

where ε_t is $N(0, 1)$ white noise. What are the corresponding asymptotic results for certain well-known recurrences in standard time series analysis?

When $|\rho| > 1$, the nonstationary first-order autoregressive process

$$X_t = \rho X_{t-1} + \sqrt{\rho^2 - 1} \varepsilon_t, \quad X_0 = 0$$

is readily shown to satisfy

$$\frac{1}{n} \ln |X_n| \rightarrow \ln |\rho| \quad \text{almost surely as } n \rightarrow \infty.$$

The quantity $\ln |\rho|$ is called the *Lyapunov exponent* of the system [3]. More precisely,

$$\mu_n = E(\ln |X_n|) = \frac{1}{2} (\ln(\rho^{2n} - 1) - \ln(2) - \gamma), \quad \sigma^2 = \text{Var}(\ln |X_n|) = \frac{\pi^2}{8}$$

where γ denotes the Euler-Mascheroni constant [4]. We wish to ascertain the distribution of the errors $(\ln |X_n| - \mu_n) / \sigma$, which do not appear to be $N(0, 1)$.

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When $|\rho| < 1$, the stationary first-order autoregressive process

$$X_t = \rho X_{t-1} + \sqrt{1 - \rho^2} \varepsilon_t$$

gives rise to a different question. Here we have

$$\mu = E(\ln |X_t|) = \frac{1}{2} (-\ln(2) - \gamma), \quad \sigma^2 = \text{Var}(\ln |X_t|) = \frac{\pi^2}{8}$$

in contrast to before. The Central Limit Theorem gives [5, 6]

$$\frac{\frac{1}{n} \sum_{t=1}^n \ln |X_t| - \mu}{\sqrt{n} \xi_\rho} \rightarrow N(0, 1) \quad \text{as } n \rightarrow \infty$$

for some constant $\xi_\rho > 0$; clearly $\xi_0 = \sigma$. What is the numerical value of ξ_ρ as a function of $\rho \neq 0$? This is our first question to be addressed.

0.1. Stationary Case. Let $f(x)$ denote the $N(0, 1)$ density function and $f^{(j)}(x)$ denote its j^{th} derivative. Since $\text{Cov}(X_1, X_{\ell+1}) = \rho^\ell$ for integer lag $\ell \geq 1$, it follows that [7]

$$\begin{aligned} E(\ln |X_1| \cdot \ln |X_{\ell+1}|) &= \sum_{j=0}^{\infty} \left| \int_{-\infty}^{\infty} \ln |x| f^{(j)}(x) dx \right|^2 \frac{\rho^{j\ell}}{j!} \\ &= \mu^2 + \sum_{k=1}^{\infty} \nu_{2k}^2 \frac{\rho^{2k\ell}}{(2k)!} \end{aligned}$$

where

$$\nu_{2k} = (-1)^{k-1} \int_{-\infty}^{\infty} \ln |x| f^{(2k)}(x) dx = 2^{k-1} (k-1)!$$

Hence

$$\begin{aligned} \text{Var} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \ln |X_t| \right) &= \frac{1}{n} \sum_{t=1}^n \sum_{s=1}^n \text{Cov}(\ln |X_t|, \ln |X_s|) \\ &= \sigma^2 + \frac{2}{n} \sum_{\ell=1}^{n-1} (n-\ell) (E(\ln |X_1| \cdot \ln |X_{\ell+1}|) - \mu^2) \\ &= \sigma^2 + \frac{2}{n} \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \sum_{\ell=1}^{n-1} (n-\ell) \rho^{2k\ell} \\ &= \sigma^2 + \frac{2}{n} \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \left(\frac{n}{1 - \rho^{2k}} - \frac{1 - \rho^{2kn}}{(1 - \rho^{2k})^2} \right) \rho^{2k} \end{aligned}$$

and therefore

$$\begin{aligned}
\xi_\rho^2 &= \lim_{n \rightarrow \infty} \text{Var} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n \ln |X_t| \right) = \sigma^2 + 2 \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \frac{\rho^{2k}}{1 - \rho^{2k}} \\
&= \frac{\pi^2}{8} + 2 \left(\frac{1}{2} \frac{\rho^2}{1 - \rho^2} + \frac{1}{6} \frac{\rho^4}{1 - \rho^4} + \frac{4}{45} \frac{\rho^6}{1 - \rho^6} + \frac{2}{35} \frac{\rho^8}{1 - \rho^8} \right. \\
&\quad + \frac{64}{1575} \frac{\rho^{10}}{1 - \rho^{10}} + \frac{64}{2079} \frac{\rho^{12}}{1 - \rho^{12}} + \frac{512}{21021} \frac{\rho^{14}}{1 - \rho^{14}} + \frac{128}{6435} \frac{\rho^{16}}{1 - \rho^{16}} \\
&\quad \left. + \frac{16384}{984555} \frac{\rho^{18}}{1 - \rho^{18}} + \frac{16384}{1154725} \frac{\rho^{20}}{1 - \rho^{20}} + \frac{131072}{10669659} \frac{\rho^{22}}{1 - \rho^{22}} + \dots \right)
\end{aligned}$$

via computer algebra. This is an example of what is called a *Lambert series* [8]. With suitably many terms, we calculate

$$\xi_{0.1} = 1.11527354305263680232\dots,$$

$$\xi_{0.3} = 1.15562165351986837602\dots,$$

$$\xi_{0.5} = 1.26199222423122947973\dots,$$

$$\xi_{0.7} = 1.52783735828651737636\dots,$$

$$\xi_{0.9} = 2.55564072887132125752\dots$$

to 20 decimal places.

As a corollary, if Y_t is an Ornstein-Uhlenbeck process (Gauss-Markov process) satisfying

$$dY_t = -\theta Y_t dt + \sqrt{2\theta} dW_t, \quad 0 \leq t \leq T$$

where $\theta > 0$ and W_t is Brownian motion with unit variance, then [5]

$$\sqrt{T} \frac{\frac{1}{T} \int_0^T \ln |Y_t| dt - \mu}{\eta_\theta} \rightarrow N(0, 1) \quad \text{as } T \rightarrow \infty$$

for some constant $\eta_\theta > 0$. A formula for η_θ is proved as follows [7]:

$$\begin{aligned}
\mathbb{E}(\ln |Y_0| \cdot \ln |Y_\ell|) &= \sum_{j=0}^{\infty} \left| \int_{-\infty}^{\infty} \ln |y| f^{(j)}(y) dy \right|^2 \frac{e^{-j\theta\ell}}{j!} \\
&= \mu^2 + \sum_{k=1}^{\infty} \nu_{2k}^2 \frac{e^{-2k\theta\ell}}{(2k)!}
\end{aligned}$$

because $\text{Cov}(Y_0, Y_\ell) = e^{-\theta\ell}$ for real lag $\ell > 0$; hence

$$\begin{aligned} \text{Var} \left(\frac{1}{\sqrt{T}} \int_0^T \ln |Y_t| dt \right) &= \frac{1}{T} \int_0^T \int_0^T \text{Cov}(\ln |Y_t|, \ln |Y_s|) ds dt \\ &= \frac{2}{T} \int_0^T \int_0^t \text{Cov}(\ln |Y_t|, \ln |Y_{t-\ell}|) d\ell dt \end{aligned}$$

upon setting $\ell = t - s$, $d\ell = -ds$ for fixed t ; hence

$$\text{Var} \left(\frac{1}{\sqrt{T}} \int_0^T \ln |Y_t| dt \right) = \frac{2}{T} \int_0^T \int_\ell^T \text{Cov}(\ln |Y_{t-\ell}|, \ln |Y_t|) dt d\ell$$

upon reversing the order of integration; hence

$$\begin{aligned} \text{Var} \left(\frac{1}{\sqrt{T}} \int_0^T \ln |Y_t| dt \right) &= \frac{2}{T} \int_0^T \int_\ell^T \text{Cov}(\ln |Y_0|, \ln |Y_\ell|) dt d\ell \\ &= \frac{2}{T} \int_0^T (T - \ell) \left(\mathbb{E}(\ln |Y_0| \cdot \ln |Y_\ell|) - \mu^2 \right) d\ell \\ &= \frac{2}{T} \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \int_0^T (T - \ell) e^{-2k\theta\ell} d\ell \\ &= \frac{2}{T} \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \left(\frac{T}{2k\theta} + \frac{e^{-2k\theta T} - 1}{(2k\theta)^2} \right) \end{aligned}$$

and therefore

$$\begin{aligned} \eta_\theta^2 &= \lim_{T \rightarrow \infty} \text{Var} \left(\frac{1}{\sqrt{T}} \int_0^T \ln |Y_t| dt \right) = \frac{1}{\theta} \sum_{k=1}^{\infty} \frac{\nu_{2k}^2}{(2k)!} \frac{1}{k} \\ &= \frac{1}{\theta} \sum_{k=1}^{\infty} \frac{2^{2k-2} (k-1)!^2}{(2k)! k} = \frac{1}{\theta} \sum_{n=0}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} (n+1)^2 (2n+1)} \\ &= \frac{1}{\theta} \left(\frac{1}{4} \pi^2 \ln(2) - \frac{7}{8} \zeta(3) \right) = \frac{1}{\theta} (0.81146307722510340753\dots)^2 \end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta function [9]. Many analogous central binomial sums appear in [10].

0.2. Nonstationary Case. Since $X_n \sim N(0, \rho^{2n} - 1)$, we deduce that

$$\begin{aligned} \mathbb{P}\left(\frac{\ln |X_n| - \mu_n}{\sigma} \leq x\right) &= \mathbb{P}\left(|X_n| \leq e^{\sigma x + \mu_n}\right) \\ &= \sqrt{\frac{2}{\pi(\rho^{2n} - 1)}} \int_0^{e^{\sigma x + \mu_n}} \exp\left(-\frac{y^2}{2(\rho^{2n} - 1)}\right) dy; \end{aligned}$$

thus

$$\begin{aligned} \frac{d}{dx} \mathbb{P}\left(\frac{\ln |X_n| - \mu_n}{\sigma} \leq x\right) &= \sqrt{\frac{2}{\pi(\rho^{2n} - 1)}} \exp\left(-\frac{e^{2(\sigma x + \mu_n)}}{2(\rho^{2n} - 1)}\right) e^{\sigma x + \mu_n} \sigma \\ &= \frac{1}{2} \sqrt{\frac{\pi}{\rho^{2n} - 1}} \exp\left(-\frac{e^{2(\sigma x + \mu_n)}}{2(\rho^{2n} - 1)} + \sigma x + \mu_n\right) \\ &= \frac{1}{2} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{1}{4} e^{2\sigma x - \gamma} + \sigma x - \frac{\gamma}{2}\right) \\ &= \frac{1}{2} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{1}{4} e^z + \frac{1}{2} z\right) \end{aligned}$$

where $z = \pi x / \sqrt{2} - \gamma$. Clearly $(\ln |X_n| - \mu_n) / \sigma$ possesses a doubly exponential density function (called a Gumbel density or Fisher-Tippett Type I extreme values density [11]) with mean = 0, variance = 1,

$$\text{skewness} = -\frac{28\sqrt{2}}{\pi^3} \zeta(3) = -1.53514159072290597506\dots$$

and kurtosis = $7 - 3 = 4$. Negativity of the third moment above confirms that the distribution is skewed to the left. Closed-form expressions for the quartiles do not exist:

$$25^{\text{th}} \text{ \%tile} = -0.45782337329420373497\dots,$$

$$\text{median} = 50^{\text{th}} \text{ \%tile} = 0.21732071404060381038\dots,$$

$$75^{\text{th}} \text{ \%tile} = 0.69796763838144042777\dots$$

but the maximum point of the density is easily found:

$$\text{mode} = \frac{\sqrt{2}(\ln(2) + \gamma)}{\pi} = 0.57186419860436852975\dots$$

It is pleasing that, upon subtracting the “trend” from an AR(1) process, such a nice residual distribution emerges (independent of both ρ and n).

As a corollary, if Y_t satisfies

$$dY_t = -\theta Y_t dt + \sqrt{-2\theta} dW_t, \quad Y_0 = 0, \quad 0 \leq t \leq T$$

where $\theta < 0$, then

$$\frac{1}{T} \ln |Y_T| \rightarrow -\theta \quad \text{almost surely as } T \rightarrow \infty$$

and the density of $(\ln |Y_T| + T\theta)/\sigma$ approaches the same doubly exponential function as before. The proof is immediate.

More generally, consider the nonstationary AR(m) process

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \cdots + a_m X_{t-m} + b \varepsilon_t,$$

$$X_0 = X_{-1} = \cdots = X_{2-m} = X_{1-m} = 0.$$

Let A denote the $m \times m$ matrix with (a_1, a_2, \dots, a_m) in the top row, 1s on the subdiagonal and 0s elsewhere. Order the complex eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ of A so that λ_1 has maximum modulus. When $|\lambda_1| > 1$, AR(m) is shown to satisfy [12]

$$\frac{1}{n} \ln |X_n| \rightarrow \ln |\lambda_1| \quad \text{almost surely as } n \rightarrow \infty.$$

This occurs, for $m = 2$, if and only if $|a_2| > 1$ or $|a_1| > 1 - a_2$. An evaluation of the residual distribution remains open.

0.3. Variations. Surely the results given in this paper are not new! A careful literature search was unsuccessful. An example in [3] inspires us to look at the stationary case with ε_t assumed to be $U(-\sqrt{3}, \sqrt{3})$ white noise. Obviously $E(X_t) = 0$ and $\text{Var}(X_t) = 1$. When $\rho = 0$, it follows that

$$E(\ln |X_t|) = \frac{1}{2} \ln(3) - 1 \approx \ln(0.637), \quad \text{Var}(\ln |X_t|) = 1$$

because each X_t is uniformly distributed.¹ When $\rho \neq 0$, this fact no longer holds and hence the relevant Central Limit Theorem parameters are not apparent.

We conclude with a recurrence that somewhat resembles Viswanath's:

$$X_t = \rho X_{t-1} \pm \sqrt{\rho^2 - 1}, \quad X_0 = 0$$

where $|\rho| > 1$ and plus/minus signs are equiprobable. While $E(X_n) = 0$ and $\text{Var}(X_n) = \rho^{2n} - 1$ as in the Gaussian nonstationary case, it seems difficult to find $E(\ln |X_n|)$ and $\text{Var}(\ln |X_n|)$, let alone to find the distribution of residuals.

¹The numerical estimate $E(\ln |X_t|) \approx \ln(0.2)$ in [3] is evidently a mistake.

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